Spectral Theory for PDEs (June 2021)

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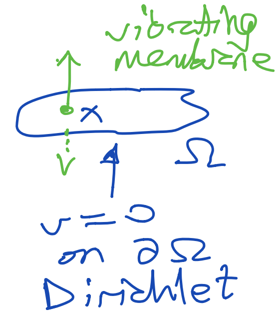
focus on discrete spectrum

eigenvalues $\lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots \to \infty$

MOTIVATION wave $v_{tt} = c^2 \Delta v$



soln.

$$v(x,t) = \sum_n \Big(\cos(\sqrt{\lambda_n}\, ct)\, a_n u_n(x) + \frac{\sin(\sqrt{\lambda_n}\, ct)}{\sqrt{\lambda_n}\, c} b_n u_n(x) \Big)$$

freq $= \sqrt{\lambda_n}\, c$

$v(x,0) = ?$
$v_t(x,0) = ?$

where

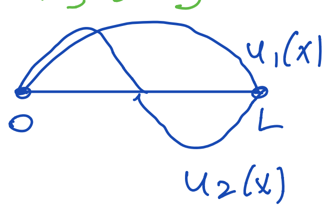
$u_n = n$-th eigenfn. of $-\Delta$ $\quad \boxed{-\Delta u_n = \lambda_n u_n}$

Q: existence of eigenfns? ONB?
formulas for eigenfns? ← $\Omega =$ interval, rectangle, disk, equilateral triangle
positivity of eigenvalues?
growth of eigenvalues as $n \to \infty$?

Chapter 2 – Examples of computable spectra

1-dim. $\quad \boxed{-u'' = \lambda u}$ for $0 < x < L$
vibrating string

$u = 0$ at $x = 0, L$



$$u_j(x) = \sin\left(\frac{j\pi x}{L}\right), \quad j = 1, 2, 3, \ldots$$

$$\lambda_j = \left(\frac{j\pi}{L}\right)^2$$ grows like j^2

scales like $\frac{1}{(\text{length})^2}$

freq. $\sqrt{\lambda_j} = \frac{j\pi}{L}$

long string $\longleftrightarrow$ low freq.

2-dim. $\boxed{-\Delta u = \lambda u}$

$$-\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = \lambda u$$

Ω = rectangle

M

L

$$u = \sin\left(\frac{j\pi x}{L}\right)\sin\left(\frac{k\pi y}{M}\right),$$

$$\lambda = \left(\frac{j\pi}{L}\right)^2 + \left(\frac{k\pi}{M}\right)^2 \Bigg\} \quad j,k = 1,2,3,\ldots$$

scaling $\sim \frac{1}{(\text{length})^2}$

Dimensional analysis $\quad \underbrace{-\Delta u}_{} = \underbrace{\lambda u}_{}$

2 derivatives scales like $\frac{1}{(\text{length})^2}$ $\quad \therefore \lambda$ must scale like $\frac{1}{(\text{length})^2}$.

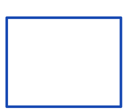
big drums $\longleftrightarrow$ low tones

Growth rate of eigenvalues?

eg $L = M = \pi \Rightarrow \lambda = j^2 + k^2$

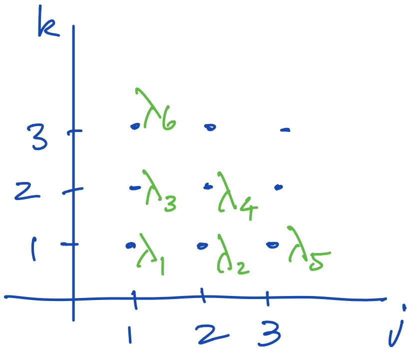
$\quad$ = square of distance from origin to point (j,k)

square



π

π



j	k	λ_n
1	1	λ_1
2	1	λ_2
1	2	λ_3
2	2	λ_4
	$\vdots$	

$\sqrt{\lambda_n}$ = radius of n-th largest lattice point

Exercise $\frac{\pi}{4} \cdot \lambda_n \sim n$ as $n \to \infty$

(for the square)

linear growth rate

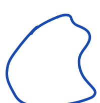
[in $\mathbb{R}^d$ growth rate $n^{2/d}$]

Weyl Law in 2-dim.

$$\lambda_n \sim \frac{4\pi n}{\text{area}(\Omega)}$$

λ_n grows like n

[area $\sim$ (length)^2]



$\Omega \subset \mathbb{R}^2$

In $\mathbb{R}^d$, $\lambda_n \sim c_d \frac{n^{2/d}}{V^{2/d}}$

Pólya Conjecture

$$\boxed{\lambda_n \geq \frac{4\pi n}{\text{area}(\Omega)}}$$

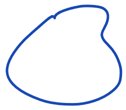
is open for $n \geq 3$

Known for special classes of domain

e.g. plane-tiling domains (triangles, parallelograms)

Analogous conj. in all dimensions.

Next: existence of eigenvalues?



$\Omega \subset \mathbb{R}^d$ $\qquad \Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_d^2}$

Method:

$$(E) \quad \begin{cases} -\Delta u = \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

$$\Rightarrow \quad -\int_\Omega (\Delta u) v \, dx = \lambda \int_\Omega u v \, dx$$

for any fn. v defined on Ω

Green's formula

$$\Rightarrow \quad \cancel{-\int_{\partial\Omega} \frac{\partial u}{\partial n} v \, dS} \to 0$$

assume $v = 0$ on $\partial\Omega$

$$+ \int_\Omega \nabla u \cdot \nabla v \, dx = \lambda \int_\Omega u v \, dx$$

$$\Rightarrow$$

$$(W) \quad \boxed{\int_\Omega \nabla u \cdot \nabla v \, dx = \lambda \int_\Omega u v \, dx}$$

for all $v \in H_0^1(\Omega)$ = Sobolev space

$v \in L^2$, $\nabla v \in L^2$

$v = 0$ on $\partial\Omega$

Define $u \in H_0^1(\Omega)$ to be a weak eigenfn. for (E) if (W) holds.

Plan

functional analysis: $\ast$ show $\exists$ ONB of weak eigenfns. $\{u_n\}$, with $\lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots \to \infty$

elliptic regularity: $\ast$ show weak eigenfns. are smooth fns.

* deduce weak eigenfns. satisfy (E)

Chapter 3 – Discrete spectral theorem

Hypotheses

Hilbert space K

" " H separable (countable dense subset)

eg $K = H^1_0(\Omega)$, $H = L^2(\Omega)$

① $K \overset{\text{dense}}{\subset} H$ with $\|u\|_H \le (\text{const.}) \|u\|_K \quad \forall u \in K$

② K is compactly imbedded in H

i.e. $\{f_n\}$ bounded in K

$\Rightarrow \exists$ subseq. $\{f_{n_m}\}$ converging in H

③ $\exists$ map $a : K \times K \to \mathbb{R}$ s.t. $a(u,v)$ is:

bilinear form

linear w.r.t. each variable

eg $a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx$

continuous

$$|a(u,v)| \le (\text{const.}) \|u\|_K \|v\|_K$$

symmetric

$$a(u,v) = a(v,u)$$

④ coercive $a(u,u) \ge (\text{const.}) \|u\|_K^2$

$$\therefore \quad c_1 \|u\|_K \underset{\text{coercive}}{\le} \sqrt{a(u,u)} \underset{\text{continuity}}{\le} c_2 \|u\|_K$$

so $a(\cdot,\cdot)$ is an inner product on K whose norm generates same topology as K-norm.

Equivalent inner

product.

Next time: we will state the theorem giving existence of the eigenvalues!

Exercises

2.1 — Find the shape of rectangle that minimizes $\lambda_1 A$ among all rectangles. (Here A = area of the rectangle.)

Shape optimization

Note: the eigenvalue is multiplied by area in this problem in order to obtain a scale-invariant quantity — the value of $\lambda_1 A$ does not change when the side lengths L and M are replaced by cL and cM.

2.2 — Guess the shape of domain that minimizes $\lambda_1 A$ among all planar domains. (Faber-Krahn theorem.)

In both these problems, we see that Nature prefers symmetric optimizers.

2.5 – Show that a square is determined by its fundamental tone, that is, by its first Dirichlet eigenvalue λ_1.

Inverse spectral theory

In other words, given a square domain and its first eigenvalue, one can determine the sidelength of the square.

2.6 – How many eigenvalues are needed to determine a rectangle?

2.7 – How many eigenvalues do you think would determine a triangle?
(Open problem! Antunes & Freitas have investigated numerically.)

Lecture 2

* Discrete spectral theorem (proof omitted)
* Variational formulas for eigenvalues
* Application to the Laplacian

Theorem 3.1 (Discrete Spectral Theorem)

Under the hypotheses above, there exist vectors $u_1, u_2, u_3, \ldots \in K$ and numbers

$$0 < \gamma_1 \leq \gamma_2 \leq \gamma_3 \leq \ldots \to \infty$$

such that:

* u_j is an eigenvector of $a(\cdot,\cdot)$ with eigenvalue γ_j, meaning

weak eigenvalue equation

$$a(u_j, v) = \gamma_j \langle u_j, v \rangle_H \qquad \forall v \in K$$

* $\{u_j\}$ is an ONB for H

* $\{u_j/\sqrt{\gamma_j}\}$ is an ONB for K w.r.t. the a-inner product.

The decomposition

$$\boxed{f = \sum_j \langle f, u_j \rangle_H u_j}$$

holds with convergence in H for each $f \in H$, and holds with convergence in K for each $f \in K$.

Idea of proof:

Show that a certain "inverse" operator associated with the bilinear form is compact and self-adjoint, and then apply the spectral theorem for compact self-adjoint operators.

(Analogy: matrix eigenvector

$$Av = \lambda v$$
$$A^{-1}v = \frac{1}{\lambda} v$$

Inverse operator has reciprocal eigenvalues!)

(eg if $a(u,v)$ corresponds to $-\Delta$ then the inverse operator $(-\Delta)^{-1}$ means the integral operator whose kernel is the Green function.)

See online notes for full details.

Chapter 4 - Variational Characterizations of Eigenvalues

Goal: To characterize the eigenvalues from Ch. 3

Motivation: how can we estimate the eigenvalues if the spectrum cannot be computed explicitly?

e.g. matrix A, real symmetric $d \times d$ matrix

$$a(u,v) = Au \cdot v = (Au)^T v = u^T A v \qquad (u, v \in \mathbb{R}^d)$$

first eigenvalue

$$\lambda_1 = \min_{v \neq 0} \frac{Av \cdot v}{v \cdot v}$$

Pf: $A = P^T D P$ diagonalization
let $Pv = w$ etc. . . . .

Rayleigh Principle for First Eigenvalue

Define Rayleigh quotient $\dfrac{a(u,u)}{\langle u,u\rangle_H}$

Theorem

First eigenvalue equals minimum of Rayleigh quotient:

$$\boxed{\gamma_1 = \min_{f \in K\setminus\{0\}} \frac{a(f,f)}{\langle f,f\rangle_H}}$$

Consequence: any choice of f will give an upper bound on first eigenvalue.

Proof: Expand $f \in K$ wrt ONB as

$$f = \sum_j c_j u_j \qquad \text{where } c_j = \langle f, u_j\rangle_H$$

This series converges in both H and K by Th. 3.1, hence

Rayleigh quotient

$$= \frac{a(f,f)}{\langle f,f\rangle_H} = \frac{\sum_j \sum_k c_j c_k\, a(u_j,u_k)}{\sum_j \sum_k c_j c_k \langle u_j, u_k\rangle_H}$$

$= \delta_{jk}$ since ONB

$$= \frac{\sum_j |c_j|^2 \gamma_j}{\sum_j |c_j|^2} \overset{(*)}{\geq} \gamma_1$$

using that $\gamma_j \geq \gamma_1$

noting that $a(u_j, u_k) = \gamma_j \langle u_j, u_k \rangle_H$ by weak eigenfn. eq. with $v = u_k$

$$= \begin{cases} \gamma_j & \text{if } j = k \\ 0 & \text{otherwise} \end{cases}.$$

Get equality in (*) if $f = u_1$ ($c_1 = 1$, $c_j = 0$ $\forall j \geq 2$).

<u>Poincaré Principle</u> ($j \geq 1$)

$$\boxed{\gamma_j = \min_{S} \max_{f \in S \setminus \{0\}} \frac{a(f,f)}{\langle f, f \rangle_H}} \qquad (P)$$

where S ranges over all <u>j-dim.</u> subspaces of K.

Proof sketch:

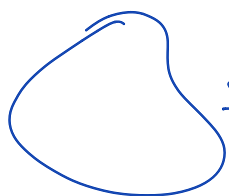
$\geq$: choose S = span of first j eigenfns.

$\leq$: choose <u>arbitrary</u> j-dim. subspace S, find vector $f \in S$ that is orthogonal to $u_1, \ldots, u_{j-1}$, then use that f in Rayleigh quotient.

<u>Comment:</u> Rayleigh & Poincaré are good for finding <u>upper</u> bounds on eigenvalues.

(computable, since enough to consider f with norm 1 in S, by homogeneity of Rayleigh quotient)

Chapter 5 — Application: discrete spectrum for the Dirichlet Laplacian



$\Omega \subset \mathbb{R}^d$

finite volume

WANT EXISTENCE FOR:

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

(separable)

Let $H = L^2(\Omega)$, $\langle u,v\rangle_{L^2} = \int_\Omega u v \, dx$

$K = H^1_0(\Omega)$, $\langle u,v\rangle_{H^1} = \int_\Omega (\nabla u \cdot \nabla v + uv)\, dx$

$H^1_0 = \{L^2$-fns. with one deriv. in L^2 and $u=0$ on $\partial\Omega\}$

Density: $C^\infty_0 \subset H^1_0 \subset L^2$

(smooth fns. with compact support)

and C^∞_0 is dense in L^2

so H^1_0 is dense in L^2.

Continuous imbedding:

$$\|u\|^2_{L^2} = \int_\Omega u^2 dx \le \int_\Omega (|\nabla u|^2 + u^2)\, dx = \|u\|^2_{H^1}$$

$\therefore$ $H^1_0 \subset L^2$ with appropriate norm bound.

Next time: finish checking hypotheses, apply Th. 3.1 and prove domain monotonicity by Rayleigh and Poincaré Principles.